Partially chaotic ensembles

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The chaotic behavior of noninvariant matrix ensembles is studied by connecting the distribution of the invariant traces of powers of the matrix with the distribution of matrix elements. Earlier results on the spacing distribution for the Gaussian orthogonal ensemble can easily be derived using the present formulation. For the ensemble in which one of the off-diagonal elements has a δ-function distribution, the spacing distribution for small spacing is derived and it is shown that it lies between that of Wigner and Poisson distributions. [S1063-651X(98)00701-6]

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I. INTRODUCTION

The matrix ensemble theory that was introduced by Wigner [1] to study the distribution of the widths and positions of compound nucleus resonances has now found many applications in different areas of physics such as condensed-matter physics and string theories. One area where matrix ensemble theory is currently used is the study of chaos. The distribution of the nearest level spacing is intimately connected with the chaotic behavior of the system. If the nearest level spacing distribution turns out to be a Wigner distribution, then the system is completely chaotic, while if it is a Poisson distribution, then it shows that the system is integrable. The main interest is to see how the level spacing distribution changes from Wigner to Poisson. This is done by studying partially chaotic ensembles.

It has been shown recently [2] that one could derive analytic expression for the spacing distribution for small dimensional matrices by first deriving an expression for the distribution of invariant traces of the matrix. In the present work we shall consider three-dimensional matrices to study the nearest level spacing distribution for two different distributions of Hamiltonian matrix elements. We describe the formulation in Sec. II and discuss the conclusions in Sec. III.

II. FORMULATION

Let us consider a three-dimensional symmetric Hamiltonian matrix with elements H_{ij} (i,j=1,2,3). The distribution of the elements H_{ij} is denoted by $F(\{H_{ij}\}), i \leq j$. We are interested in deriving an expression for the distribution of the three invariant quantities $u = \mathrm{Tr}H$, $v = \mathrm{Tr}H^2$, and $w = \mathrm{Tr}H^3$. According to the theory of probability [3], this distribution $P(u,v,w)$ is given by

$$P(u,v,w) = \int \delta(u - \mathrm{Tr}H)\,\delta(v - \mathrm{Tr}H^2) \times \delta(w - \mathrm{Tr}H^3) f(\{H_{ij}\}) \prod_{i \leq j} dH_{ij}. \quad (1)$$

To integrate over the variables H_{ij}, we first make the orthogonal transformation

$$\begin{pmatrix} H_{11} \\ H_{22} \\ H_{33} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}. \quad (2)$$

This transformation makes the first δ function $\delta(u - \sqrt{3} y_3)$ and we could immediately integrate over y_3.

Making two more transformations

$$H_{23} = \rho \cos \phi, \quad H_{13} = \rho \sin \phi$$

and

$$y_1 = R \cos \theta \quad \sqrt{2} H_{12} = R \sin \theta,$$

we can write

$$P(u,v,w) = \int \delta\left(\left(v - \frac{u^2}{3} \right) - (R^2 + y_2^2 + 2\rho^2) \right) \delta\Bigg(\left(w + \tfrac{7}{18} u^3 - \tfrac{3}{2} uv \right) - \left(\frac{1}{\sqrt{6}} y_2^3 - \frac{3}{\sqrt{6}} R^2 y_2 - \tfrac{1}{2} u y_2^2 - \tfrac{1}{2} R^2 u - u\rho^2 + \frac{3}{\sqrt{6}} \rho^2 y_2 - \frac{3}{\sqrt{2}} \rho^2 R \cos(\theta + 2\phi) \right) \Bigg) \times f \rho \, d\rho \, d\phi R \, dR \, d\theta \, dy_2. \quad (3)$$

The function f is now a function of the new variables; its final form will be given later.

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By carrying out integration over one of the angular variables and after a few simplifying steps we finally get

$$P(u,v,w)=\int \delta\left(\frac{\sqrt{6}(w+\frac{2}{9}u^3-uv)}{\left(v-\frac{u^2}{3}\right)^{3/2}}+4x_1^3-3x_1\frac{9}{2}x_3^2x_1+3\sqrt{\frac{3}{2}}x_3^2x_2\right)(f+\hat{f})d\phi\prod_{i=1}^{3}dx_i, \qquad (4)$$

where the region of integration is $\Sigma_{i=1}^3 x_i^2 \leqslant 1$. The function $f(H_{11},H_{22},H_{33},H_{12},H_{13},H_{23})$ in terms of new variables has the form

$$f\left(\frac{1}{\sqrt{2}}\left[\left(-\sqrt{\frac{3}{2}}\hat{x}_1+\frac{1}{2}\hat{x}_2\right)\cos\phi+\hat{x}_3\sin\phi\right]-\frac{1}{\sqrt{6}}\left(\frac{1}{2}\hat{x}_1+\sqrt{\frac{3}{2}}\hat{x}_2\right)+\frac{1}{3}u,-\frac{1}{\sqrt{2}}\left[\left(-\sqrt{\frac{3}{2}}\hat{x}_1+\frac{1}{2}\hat{x}_2\right)\cos\phi+\hat{x}_3\sin\phi\right]-\frac{1}{\sqrt{6}}\left(\frac{1}{2}\hat{x}_1+\sqrt{\frac{3}{2}}\hat{x}_2\right)+\frac{1}{3}u,\frac{2}{\sqrt{6}}\left(\frac{1}{2}\hat{x}_1+\sqrt{\frac{3}{2}}\hat{x}_2\right)+\frac{1}{3}u,\frac{1}{\sqrt{2}}\left[\hat{x}_3\cos\phi-\left(-\sqrt{\frac{3}{2}}\hat{x}_1+\frac{1}{2}\hat{x}_2\right)\sin\phi\right],\frac{1}{\sqrt{2}}\rho\sin\frac{\phi}{2},\frac{1}{\sqrt{2}}\rho\cos\frac{\phi}{2}\right) \qquad (5)$$

and $\hat{f}$ is obtained from f by replacing ϕ with $\phi+2\pi$; $\hat{x}_i=(v-u^2/3)^{1/2}x_i$ and $\rho=[(v-u^2/3)(1-\Sigma_{i=1}^3 x_i^2)]^{1/2}$.

Before we proceed further we apply Eq. (4) to the rotationally invariant Gaussian orthogonal ensemble (GOE) for which

$$f(\{H_{ij}\})=\exp(-\mathrm{Tr}H^2). \qquad (6)$$

It is easy to see from Eq. (5) that the function f in terms of the new variables is

$$f=\exp(-v). \qquad (7)$$

Thus, from Eq. (4), $P(u,v,w)$ for the GOE is given by

$$P(u,v,w)=[\exp(-v)]\int \delta\left(\frac{\sqrt{6}(w+\frac{2}{9}u^3-uv)}{\left(v-\frac{u^2}{3}\right)^{3/2}}+4x_1^3-3x_1+\frac{9}{2}x_3^2x_1+\frac{3\sqrt{3}}{2}x_3^2x_2\right)\prod_{i=1}^{3}dx_i. \qquad (8)$$

It is shown in the Appendix that δ function in Eq. (8) implies that

$$-1\leqslant\frac{\sqrt{6}(w+\frac{2}{9}u^3-uv)}{\left(v-\frac{u^2}{3}\right)^{3/2}}\leqslant 1. \qquad (9)$$

Therefore, $P(u,v,w)$ for the GOE is given by

$$P(u,v,w)=\exp(-v), \qquad (10)$$

the range of w being given by Eq. (9), $-\sqrt{3v}\leqslant u\leqslant\sqrt{3v}$, and $0\leqslant v\leqslant\infty$.

In the earlier work [2] on two dimensions it was shown that if the functional form of the distribution f is noninvariant under rotation, then the nearest spacing distribution starts deviating from Wigner's spacing distribution. We would now like to introduce a noninvariant distribution f in three dimensions and see how the spacing distribution behaves for small spacing s. One such simple distribution introduced by Molinari and Sokolov [4] is

$$f=[\exp(-\mathrm{Tr}H^2)]\delta(H_{13}), \qquad (11)$$

in which the off-diagonal element H_{13} is taken to be zero. Using Eqs. (4) and (5), we find that $P(u,v,w)$ for this noninvariant ensemble is given by

$$P(u,v,w)=\int \delta\left(\frac{\sqrt{6}(w+\frac{2}{9}u^3-uv)}{\left(v-\frac{u^2}{3}\right)^{3/2}}+4x_1^3-3x_1+\frac{9}{2}x_3^2x_1+\frac{3\sqrt{3}}{2}x_3^2x_2\right)\left[\left(v-\frac{u^2}{3}\right)\left(1-\sum_{i=1}^{3}x_i^2\right)\right]^{-1/2}\times[\exp(-v)]\prod_{i=1}^{3}dx_i. \qquad (12)$$

Since we are interested in the spacing distribution, we first write the distribution of the eigenvalues E_i. This is related to $P(u,v,w)$ by

$$g(E_1,E_2,E_3)=\int \delta\left(\sum_{i=1}^{3}E_i-u\right)\delta\left(\sum_{i=1}^{3}E_i^2-v\right)\times\delta\left(\sum_{i=1}^{3}E_i^3-w\right)\prod_{i<j}|E_i-E_j|\times P(u,v,w)du\,dv\,dw. \qquad (13)$$

According to Wigner [3], the nearest level spacing distribution $p(s)$ can be obtained by setting $E_1=-s/2$ and $E_2=s/2$ and integrating over E_3 from $-\infty$ to $-s/2$ and $s/2$ to ∞ in Eq. (13). Thus $p(s)$ is given by

$$p(s)=s\ \exp\left(-\frac{s^2}{2}\right)\int_{s/2}^{\infty}\frac{\left(u^2-\frac{s^2}{4}\right)}{\left(\frac{2}{3}u^2+\frac{s^2}{2}\right)^{1/2}}\exp(-u^2)$$

$$\times\delta\left(\frac{\sqrt{6}(\frac{2}{9}u^3-\frac{1}{2}s^2u)}{(\frac{2}{3}u^2+\frac{1}{2}s^2)^{3/2}}+\left(4x_1^3-3x_1+\frac{9}{2}x_3^2x_1\right.\right.$$

$$\left.\left.+\frac{3\sqrt{3}}{2}x_3^2x_2\right)\right)\left[1-\sum_{i=1}^{3}x_1^2\right]^{-1/2}du\prod_{i=1}^{3}dx_i. \quad (14)$$

We can easily carry out the integration over x_3 using the δ function to get

$$p(s)=s\ \exp\left(-\frac{s^2}{2}\right)\int_{s/2}^{\infty}\frac{u^2-\frac{s^2}{4}}{\left(\frac{2}{3}u^2+\frac{s^2}{2}\right)^{1/2}}\exp(-u^2)$$

$$\times[\lambda+4x_1^3-3x_1]^{-1/2}\left[-\lambda-4x_1^3+3x_1\right.$$

$$\left.-[1-(x_1^2+x_2^2)]\left(\frac{9}{2}x_1+\frac{3\sqrt{3}}{2}x_2\right)\right]^{-1/2}du\ dx_1\ dx_2, \quad (15)$$

where

$$\lambda=\frac{\sqrt{6}(\frac{2}{9}u^3-\frac{1}{2}s^2u)}{(\frac{2}{3}u^2+\frac{1}{2}s^2)^{3/2}}. \quad (16)$$

The region of integration over x_1,x_2 is the one for which the product of the terms in square brackets in Eq. (15) remains positive.

By making a change of variable in x_2 and after some simplification, we can write Eq. (16) as

$$p(s)=s\ \exp\left(-\frac{s^2}{2}\right)\int_{s/2}^{\infty}\frac{\left(u^2-\frac{s^2}{4}\right)}{\left(\frac{2}{3}u^2+\frac{s^2}{2}\right)^{1/2}}\exp(-u^2)$$

$$\times[\lambda+4x_1^3-3x_1]^{-1/2}[-\lambda+4q^3$$

$$-3q]^{-1/2}du\ dx_1\ dq. \quad (17)$$

In writing the expressions for various probability distributions above, we have omitted any numerical constants that multiply these expressions, as they can be obtained in the end by normalization condition.

In general, it is difficult to integrate Eq. (17) in closed form. Since we are interested in the behavior of $p(s)$ for small s, we first write λ as

$$\lambda=1-\frac{27}{8}\frac{s^2}{u^2}. \quad (18)$$

We can then find the three roots of the cubic in this approximation and carry out integrations over x_1,q for small s, which gives

$$(4\ \ln 2-\ln s+\ln u). \quad (19)$$

In the small-s limit, carrying out the final integration over u then gives

$$p(s)=\text{const}\times s(4\ \ln 2-\tfrac{1}{2}\gamma-\ln s),\quad s\to 0, \quad (20)$$

where γ is Euler's constant [5].

Comparing Eq. (20) with the usual Wigner spacing distribution $p_W(s)\to\text{const}\times s$, we find that $p(s)$ for partially chaotic ensembles is of the form $-s\ \ln s$ for small s, that is, it has a logarithmic divergence for small s.

We would like to convert Eq. (20) into a form that is usually used in the study of chaos. Writing $1-\ln s/(4\ \ln 2-\frac{1}{2}\gamma)$ approximately as $\exp[-\ln s/(4\ \ln 2-\frac{1}{2}\gamma)]$, we get

$$p(s)=\text{const}\times s^{0.6} \quad (21)$$

for small values of s. The approximate form given by Eq. (21) will be valid as long as $0.09\leqslant s\ll 1$. Thus the nearest level spacing distribution for the noninvariant distribution is in between that of the Wigner and Poisson distributions.

III. CONCLUDING REMARKS

In the present work we have developed a formulation to study the nearest level spacing distribution by connecting it directly to the distribution of Hamiltonian matrix elements. This is done by first deriving an expression for the distribution of the invariant traces of powers of H. It can be easily shown that the earlier result for the spacing distribution of the invariant GOE [3] can be obtained from Eqs. (10) and (13). The present formulation provides a way to study the spacing distribution for any noninvariant Hamiltonian matrix elements distribution. In the present work it is applied to a noninvariant distribution in which one of the off-diagonal elements has a δ-function distribution.

In the present study we have considered 3×3 random matrices and obtained analytic results. In the past Cheon [7] has carried out numerical calculations for large dimensional matrices to see the transition of the spacing distribution from Wigner to Poisson. One interesting result is that the spacing distribution in these calculations can be fitted by the phenomenological Brody [8] distribution. The Brody distribution for small values of s has the same behavior as the one given by Eq. (21). Thus our results are consistent with the one obtained by Cheon [7] for large dimensional matrices.

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APPENDIX

To find the moments of the δ function in expression (8) we set

$$t=\frac{\sqrt{6}(w+\frac{2}{9}u^3-uv)}{\left(v-\frac{u^2}{3}\right)^{3/2}};$$

then we can write a moment generating function [6] of the δ function appearing in Eq. (8) as

$$\sum{}_{n=0}^{\infty}\frac{(-i\alpha)^n}{n!}\int dt\ t^n\delta\left(t+4x_1^3-3x_1+\tfrac{9}{2}x_3^2x_1+\frac{3\sqrt{3}}{2}x_3^2x_2\right)\prod_{i=1}^{3}dx_i$$

$$=\int \exp(-i\alpha)\left[4x_1^3-3x_1+\tfrac{9}{2}x_3^2x_1+\frac{3\sqrt{3}}{2}x_3^2x_2\right]\prod_{i=1}^{3}dx_i, \tag{A1}$$

the integration being over the region $\Sigma_{i=1}^{3}x_i^2\leqslant 1$. Expanding the exponential function and carrying out the integrations, one can write

$$\sum_{n=0}^{\infty}\frac{(-i\alpha)^n}{n!}\langle t^n\rangle=\sum_{n=0}^{\infty}(-i\alpha)^{2n}\sum_{s,\mu}{}_2F_1(2\mu+\tfrac{1}{2},-2n+s+2\mu;n+s+2\mu+\tfrac{5}{2};\tfrac{3}{2})$$

$$\times\frac{(-1)^s}{s!}\frac{2^{2s-4\mu-2}3^{2n-s+\mu+1}}{(2n-s-2\mu)!}\frac{\Gamma(2\mu+\frac{1}{2})}{\sqrt{\pi}\mu!}\frac{\Gamma(n+s-\mu+\frac{1}{2})}{\Gamma(n+s+2\mu+\frac{5}{2})}, \tag{A2}$$

where ${}_2F_1$ is a hypergeometric function [5]. Thus all the odd moments of t vanish and even moments are $\langle t^2\rangle=\frac{1}{3}$, $\langle t^4\rangle=\frac{1}{5}$, etc., which shows that t has a uniform distribution between -1 and $+1$.

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